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Power Spectrum in Krein Space Quantization

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Abstract The power spectrum of scalar field and space-time metric perturbations produced in the process of inflation of universe, have been presented in this paper by an alternative approach to field quantization namely, Krein space quantization (Gazeau et al. in Class. Quantum Gravity 17:1415, 2000; Takook in Int. J. Mod. Phys. E 11:509, 2002; Rouhani and Takook in gr-gc/0607027). Auxiliary negative norm states, the modes of which do not interact with the physical world, have been utilized in this method. Presence of negative norm states play the role of an automatic renormalization device for the theory.

Keywords Power spectrum · Krein space quantization · de Setter space

1 Introduction

The minimally coupled scalar field in de Sitter space plays an important role in the inflationary model as well as in the linear quantum gravity [3, 4]. As proved by Allen [1], a covariant quantization of minimally coupled scalar field cannot be constructed by positive norm states alone. To cope with this problem the Krein space quantization has been utilized [5]. Preserving the covariance principle and ignoring the positivity condition, we have performed the field quantization in the Krein space, a combined Hilbert and anti-Hilbert space [11]. This process resembles Gupta-Bleuler quantization of the electrodynamic equations in Minkowski space. It has been proven that the use of the two sets of solutions (positive and negative norms states) is a necessary feature for preservation of (1) causality (locality), (2) covariance, and (3) elimination of the infrared divergence for the minimally coupled scalar field in de Sitter space [5, 18]. The ultraviolet divergence in the stress tensor disappears as well, in other words the quantum free scalar field in this method is automatically

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renormalized. The effect of "unphysical" states (negative norm states) appears in the above theory as a natural renormalization mechanism.

Following above studies and through the same new approach, the spectrum of density perturbation produced during inflation, has been calculated in this paper. In the next section we briefly recall definition of the spectrum of density fluctuation. Sections 3 is devoted to calculation of the spectrum of scalar field in Krein space quantization. The spectrum of metric perturbation is calculated in Sect. 4. Brief conclusion and outlook are given in conclusion.

2 Notation

In the linear schema of evolution of universe, the density perturbation is presented to be a function governed by simple stochastic properties. For a Gaussian distribution these are specified completely by the spectrum of density fluctuation. The cosmologists are interested in the density contrast and the quantity R which measures the associated spatial curvature perturbation. Let f denote any one of these perturbations. Its most important property is its spectrum, which is essentially the smoothed modulus-squared of its Fourier coefficient. To be precise, the spectrum may be defined as the quantity [8]

$$P_f \equiv (Lk/2\pi)^3 4\pi \langle |f_{\mathbf{k}}|^2 \rangle, \tag{1}$$

where L is the size of the box for the Fourier expansion, and the bracket denotes the average over a small region of k-space. The normalization is chosen to give a simple formula for the dispersion (the root mean square vacuum fluctuation) of f, which we shall denote by σ_f. From the Fourier expansion one has $\sigma_f^2 = \sum \langle |f_{\mathbf{k}}|^2 \rangle$, which lead to [9]

$$\sigma_f^2 \equiv \langle f^2(\mathbf{x}) \rangle = \int_0^\infty P_f(k) \frac{dk}{k}. \tag{2}$$

P_f is the spectrum, and the bracket now denoting the spatial average.

We briefly recall the quantization of the massless scalar field $\phi_p(x)$ in de Sitter space in the following coordinates

$$ds^2 = g_{\mu\nu}^{dS} dx^\mu dx^\nu = dt^2 - a^2(t) d\mathbf{x}^2 = dt^2 - e^{2Ht} d\mathbf{x}^2, \tag{3}$$

where $a(t)$ is a scale factor and $H = \frac{\dot{a}}{a}$ is the Hubble parameter. The scalar field operator $\phi_p(x)$ can be represented in the form

$$\phi_p(x) = (2\pi)^{-3/2} \int d^3k [c(k) u_k(t) e^{i\mathbf{k}.\mathbf{x}} + c^\dagger(k) u_k^*(t) e^{-i\mathbf{k}.\mathbf{x}}], \tag{4}$$

where $u_k(t)$ satisfies the equation [13]

$$\ddot{u}_k + 3H\dot{u}_k + \frac{k^2}{a^2} u_k = 0. \tag{5}$$

The exact solutions of the field equation are [10, 16]

$$u_{k,p} = \frac{e^{-ik\tau}}{a\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right), \qquad u_{k,n} = \frac{e^{+ik\tau}}{a\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right), \tag{6}$$

where modes $u_{k,p}$ ($u_{k,n}$) represent positive (negative) norm states and τ is conformal time $\frac{1}{aH} = -\tau$. Then the quantity $\langle\phi_p^2\rangle$ may be simply expressed in terms of $u_{k,p}$ of the above solution:

$$\langle\phi_p^2\rangle = \frac{1}{(2\pi)^3}\int |u_{k,p}|^2 d^3k = \frac{1}{(2\pi)^3}\int \frac{d^3k}{k}\left(\frac{1}{2a^2} + \frac{H^2}{2k^2}\right). \tag{7}$$

The first term is the usual contribution from vacuum fluctuations in Minkowski space. This contribution can be eliminated by renormalization. Accordingly the spectrum of the scalar field, defined by (2), after the renormalization of the first term of the (7), is given by

$$P_\phi(k) = \frac{H^2}{4\pi^2}. \tag{8}$$

The metric perturbation spectrum is also [16]

$$P_\Re = \frac{H^4}{4\pi^2\dot{\phi}^2}. \tag{9}$$

3 Spectrum of Scalar Field in Krein Space

In this section we calculate the spectrum of scalar field perturbations in Krein space quantization thoroughly. First, we briefly recall the Krein space quantization. In the previous paper [5, 15, 19], we present the free field operator in the Krein space quantization. The field operator in Krein space is build by joining two possible solutions of field equations, positive and negative frequency states

$$\phi(x) = \frac{1}{2}[\phi_p(x) + \phi_n(x)], \tag{10}$$

where

$$\phi_p(x) = (2\pi)^{-3/2}\int d^3k[c(k)u_{k,p}(t)e^{i\mathbf{k.x}} + c^\dagger(k)u^*_{k,p}(t)e^{-i\mathbf{k.x}}],$$

$$\phi_n(x) = (2\pi)^{-3/2}\int d^3k[b(k)u_{k,n}(t)e^{-i\mathbf{k.x}} + b^\dagger(k)u^*_{k,n}(t)e^{i\mathbf{k.x}}].$$

$c(k)$ and $b(k)$ are two independent operators. Creation and annihilation operators are constrained to obey the following commutation rules

$$[c(k), c^\dagger(k')] = \delta(k-k'), \qquad [b(k), b^\dagger(k')] = -\delta(k-k'). \tag{11}$$

The other commutation relations are all equal to zero. The vacuum state $|\Omega\rangle$ is then defined by

$$c^\dagger(k)|\Omega\rangle = |1_k\rangle, \qquad c(k)|\Omega\rangle = 0, \tag{12}$$

$$b^\dagger(k)|\Omega\rangle = |\bar{1}_k\rangle, \qquad b(k)|\Omega\rangle = 0, \tag{13}$$

where $|1_k\rangle$ is called a one particle state and $|\bar{1}_k\rangle$ is called a one "un-physical state".

By imposing the physical interaction on the field operator, only the positive norm states are affected. The negative modes do not interact with the physical states or real physical world, thus they can not be affected by the physical interaction as well. The auxiliary negative norm states in our problem are similar to the ghost states in the standard gauge QFT. In the gauge QFT, the auxiliary negative norm states (ghost states), can neither propagate in the physical world nor interact with physical states. In the previous works, we have shown that presence of negative norm states play the role of an automatic renormalization device for certain problems [5, 7, 14, 15, 17, 19, 20].

In the Minkowski space, the ultraviolet divergence of the vacuum energy of the quantum field is removed by the normal ordering. In the curved space-time, the standard renormalization of the ultraviolet divergence of the vacuum energy is accomplished by subtracting the local divergencies of Minkowski space ((4.5) in [2]),

$$\langle\Omega| : T_{\mu\nu} : |\Omega\rangle = \langle\Omega|T_{\mu\nu}|\Omega\rangle - \langle 0|T_{\mu\nu}|0\rangle,$$

$|\Omega\rangle$ is the vacuum state in curved space and $|0\rangle$ is vacuum state in Minkowski space. The mines sign in the above equation can be interpreted as the negative norm states which is added to the field operator. This interpretation resembles Krein space quantization where the negative norm states are considered as well. These negative norm states however, are defined in Minkowski space alone and they are not the solution of the wave equation in the curved space time. In other words this renormalization vividly breaks the symmetry of the curved space.

There are other possibilities for removing the local divergencies in the curved space-the so called non-standard renormalization schemata-the local divergencies of curved space is removed by the quantities defined in the very same curved space-time. In this case, the mines sign can be interpreted as the negative norm states which is added to the field operator and they are the solutions of the wave equation in the curved space-time. This scheme seams to be far more logical since the curved space symmetry has been preserved after renormalization procedure. In the previous or "standard" renormalization procedure the vacuum is defined globally while the singularities are removed locally!

An alternative interpretation of above method is as follows. In the curved space-time ($g_{\mu\nu}$), the negative norm states could be constructed based upon two different perspectives, which are defined with respect to the choice of the space-time background field ($g^{BG}_{\mu\nu}$)

$$g_{\mu\nu} = g^{BG}_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + h^{Mi}_{\mu\nu} = g^{dS}_{\mu\nu} + h^{dS}_{\mu\nu},$$

where $h^{Mi}_{\mu\nu}$ and $h^{dS}_{\mu\nu}$ are perturbation on the Minkowskian and de Sitter background respectively. In the first perspective the background metric is the Minkowskian $g^{BG}_{\mu\nu} = \eta_{\mu\nu}$. In order to confirm the equivalence of the results of the previous works with findings of this paper, we have considered the negative norm states in the Minkowskian space. In this case, the physical result is similar to the renormalization of the QFT in curved space-time with respect to the Minkowskian space-time [2]. In the second perspective the background metric is the de Sitter space, $g^{BG}_{\mu\nu} = g^{dS}_{\mu\nu}$, and the physical result is similar to the non-standard renormalization of the QFT in curved space-time with respect to the de Sitter space-time. The negative norms are defined with respect to the de Sitter background [5]. These perspectives lead to construction of two independent field operators.

3.1 First Perspective

The negative norms are defined with respect to the Minkowskian background $\eta_{\mu\nu}$. The equations of the negative norm states and their solutions are:

$$\ddot{v}_k + \frac{k^2}{a^2} v_k = 0, \qquad v_{k,n} = \frac{e^{+ik\tau}}{a\sqrt{2k}}. \tag{14}$$

The scalar field operator in such a perspective (through Krein quantization method) can be written as:

$$\phi(x) = (2\pi)^{-3/2} \int d^3k\{[c(k)u_{k,p}(t)e^{+i\mathbf{k}.\mathbf{x}} + c^\dagger(k)u^*_{k,p}(t)e^{-i\mathbf{k}.\mathbf{x}}] + [b(k)v_{k,n}(t)e^{-i\mathbf{k}.\mathbf{x}} + b^\dagger(k)v^*_{k,n}(t)e^{i\mathbf{k}.\mathbf{x}}]\}. \tag{15}$$

Then quantity $\langle\phi^2\rangle$ may be simply expressed in terms of above solutions:

$$\begin{aligned}\langle\phi^2\rangle &= \frac{1}{(2\pi)^3}\int |u_{k,p}|^2 d^3k - \frac{1}{(2\pi)^3}\int |v_{k,n}|^2 d^3k \\ &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{k}\left(\frac{1}{2a^2} + \frac{H^2}{2k^2}\right) - \frac{1}{(2\pi)^3}\int \frac{d^3k}{2ka^2} = \frac{1}{(2\pi)^3}\int \frac{d^3k}{k}\left(\frac{H^2}{2k^2}\right).\end{aligned} \tag{16}$$

Accordingly the spectrum, defined by (2), is given by

$$P_\phi = \frac{H^2}{4\pi^2}. \tag{17}$$

This result is same as (8).

3.2 Second Perspective

The negative norm states are defined with respect to the curved space-time background $g^{dS}_{\mu\nu}$. The scalar field operator in this perspective (through Krein quantization method) could be written as [5]:

$$\phi(x) = (2\pi)^{-3/2} \int d^3k\{[c(k)u_{k,p}(t)e^{i\mathbf{k}.\mathbf{x}} + c^\dagger(k)u^*_{k,p}(t)e^{-i\mathbf{k}.\mathbf{x}}] + [b(k)u_{k,n}(t)e^{-i\mathbf{k}.\mathbf{x}} + b^\dagger(k)u^*_{k,n}(t)e^{i\mathbf{k}.\mathbf{x}}]\}, \tag{18}$$

where $u_{k,p}$ and $u_{k,n}$ are defined in (6). The two-point function for the above field operator [18]:

$$\mathcal{W}(x,x') = \frac{iH^2}{8\pi}\epsilon(t-t')[\delta(1-\mathcal{Z}(x,x')) - \theta(\mathcal{Z}(x,x')-1)], \tag{19}$$

where

$$\epsilon(x^0 - x'^0) = \begin{cases} 1, & t > t', \\ 0, & t = t', \\ -1, & t < t', \end{cases}$$

and θ is the Heaviside step function. $\mathcal{Z}(x,x') = \cosh H\sigma$ and σ is the invariant geodesic distance between the points x and x'. It is clear that the propagator is well defined and free of any divergencies (apart from the delta function that does not appear in the present calculation any way). Then quantity $\langle\phi^2(x)\rangle$ may be simply expressed in terms of the time-ordered product two-point function in coinciding points [20]

$$\langle\phi^2(x)\rangle = \langle 0|T\phi(x)\phi(x)|0\rangle = iG_T(x-x) = i\Re G_F(x,x). \tag{20}$$

G_F is usual Feynman propagator ((9.52) in [2])

$$\Re G_F(x,x) \approx \frac{-a_1(x)}{16\pi} = \frac{H^2}{8\pi}. \tag{21}$$

Using (19) and (20), we obtain

$$\langle\phi^2\rangle = \frac{iH^2}{8\pi}, \qquad \langle|\phi|^2\rangle = \frac{H^2}{8\pi}. \tag{22}$$

Accordingly the spectrum, defined by (2), easily is given by:

$$P_\phi = \frac{H}{4\pi^2} k e^{-\alpha k^2}, \tag{23}$$

where $\alpha = \frac{1}{\pi H^2}$. It is clear that the scale invariance is broken here. This is similar to the previous works where the scale invariance is broken due to consideration of perturbation of the gravitational field [6].

It is important to note that in the second perspective a novel cutoff (the exponent) appears which is related to the characteristic dS scale α. This significant difference stems from the choice of dS background as opposed to the Minkowskian counterpart.

4 Spectrum of Space-Time Metric Perturbation

The linear theory of cosmological perturbations (namely the scalar field and gravitational field perturbation considered here) represent a cornerstone of modern cosmology and is used to describe the formation and evolution of structures in the universe. The seeds of these inhomogeneities were generated during inflation and stretched over astrophysical because of a phase of rapid expansion the universe. Study of inflation is special from the point of view of perturbations. Any perturbation in the scalar field means a perturbation to the energy-momentum tensor, $\delta\phi \Rightarrow \delta T_{\mu\nu}$. A perturbation in the energy-momentum tensor implies through the Einstein's equation of motion, a perturbation of the metric

$$\left[\delta R_{\mu\nu} - \frac{1}{2}\delta(g_{\mu\nu}R)\right] = 8\pi G\delta T_{\mu\nu} \quad \Longrightarrow \quad \delta g_{\mu\nu}. \tag{24}$$

Therefore we conclude that the perturbations of the scalar field and that of the metric are tightly coupled to each other and have to be studied together. The spectrum of the metric perturbations is [9, 12]

$$P_\Re^{1/2} = \frac{H}{\dot\phi} P_\phi^{1/2}. \tag{25}$$

Therefore, the power spectrum for metric perturbations at horizon scale in the first perspective is given by

$$P_\Re = \frac{H^4}{4\pi^2\dot\phi^2}, \tag{26}$$

and in the second perspective is

$$P_{\Re} \cong \frac{H^3}{4\pi^2\dot{\phi}^2} k e^{-\alpha k^2}. \tag{27}$$

Similar to the previous case, a dS-induced cutoff (the exponent) appears which is connected with the characteristic dS scale α. The emergence of a cutoff in this result is completely new. The physical reason for the different behavior of the spectra in these two perspectives is the breaking of de Sitter invariance in the first perspective [1] and its preservation in the second [5]. One of the physical implications of such a "cutoff" is the calculation of the CMBR anisotropy, which will be considered in the forthcoming paper.

5 Conclusion

The negative frequency solutions of the field equations are needed for the covariant quantization in the minimally coupled scalar fields in de Sitter space. Contrary to Minkowski space, the elimination of de Sitter negative norms in this case breaks the de Sitter invariance. In other words, to restore the de Sitter invariance, one needs to take into account the negative norm states i.e. the Krein space quantization. This provides a natural tool for renormalization of the theory [5]. In the present paper, the spectrum of the scalar field perturbations, has been calculated through the Krein space quantization. It is found that the theory is automatically renormalized. The negative norm states could be constructed based upon two different perspectives. In the first case negative norm states could be defined in flat space-time similar to renormalization of the QFT in curved space-time [2]. The standard result is obtained in this case, (8) and (9). In the second perspective, the negative norm states are defined with respect to the curved space-time. In this case the scale invariance is broken. The scale invariance is also broken due to consideration of perturbation of the interaction field [6].

In the second perspective a dS-induced cutoff appears which is connected with the characteristic dS scale α in the first scheme this cutoff is missing. In other words the global properties of the Minkowski and the dS backgrounds do result in this specific departure. The different behavior of the spectra in these two perspectives is due to the breaking of de Sitter invariance in the first perspective and its preservation in the second. In the forthcoming paper the effect of this spectrum on CMBR anisotropy will be discussed in detail.

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